

# Lê cycles and Milnor classes\*

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## Abstract

The purpose of this work is to establish a link between the theory of Chern classes for singular varieties and the geometry of the varieties in question. Namely, we show that if  $Z$  is a hypersurface in a compact complex manifold, defined by the zero-scheme of a nonzero holomorphic section of a very ample line bundle, then its Milnor classes, regarded as elements in the Chow group of  $Z$ , determine the global Lê cycles of  $Z$ ; and viceversa: The Lê cycles determine the Milnor classes. Morally this implies, among other things, that the Milnor classes determine the topology of the local Milnor fibres at each point of  $Z$ , and the geometry of the local Milnor fibres determines the corresponding Milnor classes.

## Introduction

In this article, a hypersurface means a codimension one submanifold of a complex manifold. We consider compact hypersurfaces defined by the zero-scheme of a nonzero holomorphic section of a very ample line bundle, and establish a deep relation between their Milnor classes and their global Lê cycles: We exhibit each Milnor class as an explicit polynomial in the Lê cycles and viceversa.

To explain what this means, recall first that Chern classes of complex manifolds have played for decades a major role in complex geometry and topology. There are several extensions of this important concept for singular varieties, each having its own interest and characteristics. Perhaps the most interesting of these are the Schwartz-MacPherson and the Fulton-Johnson classes. Milnor classes are global invariants of singular hypersurfaces in compact complex manifolds, elements in the Chow group of the singular variety, which measure the difference between the Schwartz-MacPherson and the Fulton-Johnson classes.

The literature about Milnor classes, and more generally about Chern classes for singular varieties, is large. And yet, the study of the geometric and topological information that these classes encode is a branch of mathematics which

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still is in its childhood. Contributing towards that goal is precisely the purpose of this article.

Lê cycles are analytic cycles that describe, among other things, the topology of the local Milnor fibres: We know from [22, 23] that there is a Lê cycle in each dimension, from 0 to that of the singular set, and the multiplicity of the Lê cycles at each point says how many handles of the corresponding dimension we must attach to a ball in order to construct the local Milnor fibre (up to homeomorphism).

Lê cycles were introduced by D. Massey for holomorphic map-germs by means of polar varieties. An interpretation of these cycles was given in [16] in terms of the Chern class of the tautological line bundle corresponding to the divisor determined by blow up along the singular variety. Using [16], the concept of Lê cycles extends naturally to global singular hypersurfaces in compact complex manifolds. These are elements in the Chow group, whose restriction to a neighborhood of each point are the classes of the usual Lê cycles.

Milnor classes were originally defined as elements in the singular homology group of the hypersurface (or a complete intersection). It was observed in the work of W. Fulton and K. Johnson ([15]) and G. Kennedy ([19]) that the Fulton-Johnson classes and the Schwartz-MacPherson classes, and hence the Milnor classes, can actually be considered as cycles in the corresponding Chow group. Milnor classes have support in the singular set of the variety, and there is a Milnor class in each complex dimension, from 0 up to the dimension of the singular set.

Similarly, Lê cycles also have support in the singular set of the hypersurface and there is a Lê cycle in each dimension, from 0 up to that of this singular set. This makes it natural to ask what is the relation among the global Lê cycles and the Milnor classes, and that is the subject we explore in this article.

The concept of Milnor classes appeared first in P. Aluffi's work [1, 2] under the name of  $\mu$ -class. Milnor classes also appear implicitly in A. Parusiński and P. Pragacz' article [25]. The actual name of Milnor classes was coined later by various authors at about the same time (see [9, 10, 36, 26]); the name comes from the fact that when the singularities of  $Z$  are all isolated, the Milnor class in dimension 0, which is an integer, is the sum of the local Milnor numbers (by [33]), while all other Milnor classes are 0 (by [34]).

There are interesting recent articles about Milnor classes by various authors, as for instance P. Aluffi, J.-P. Brasselet, J. Schürmann, M. Tibăr, S. Yokura, T. Ohmoto, L. Maxim and others. There are also important generalisations of this concept to different settings (see for instance [2, 12, 37]; also [7]). Even so, Milnor classes are still somehow mysterious objects, even "esoteric".

Milnor classes appear for instance in [4] in relation with Feynman graphs in perturbative quantum field theory. They appear also in [5] in relation with Donaldson-Thomas type invariants for Calabi-Yau threefolds, via the classes introduced by P. Aluffi in [3] for possibly nonreduced schemes. In the case of a hypersurface in a complex manifold, the Aluffi classes are used in [3] to give an interpretation of the Milnor classes.

The aim of this note is two-folded: On the one hand we use Milnor classes

to get global information about L  cycles; and on the other hand we use L  cycles to get geometric and topological information about Milnor classes. Of course this provides also a description of the Aluffi classes in terms of L  cycles (Corollary 5.7) in the case of hypersurfaces in complex manifolds.

The main result in this article is the following theorem:

**Theorem 1.** *Let  $M$  be a compact complex variety and  $L$  a very ample line bundle on  $M$ . Consider the zero-scheme  $Z$  of a nonzero holomorphic section  $s$  of  $L$  and denote by  $Z_{\text{sing}}$  the singular set of  $Z$ . Then the Milnor classes  $\mathcal{M}_s(Z)$  and the L  cycles  $\Lambda_s(Z)$  of  $Z$ ,  $s = 0, \dots, d$  where  $d = \dim(Z_{\text{sing}})$ , determine each other by the formulas:*

$$\mathcal{M}_k(Z) = \sum_{l=0}^{d-k} (-1)^{l+k} \binom{l+k}{k} c_1(L|_Z)^l \cap \Lambda_{l+k}(Z),$$

and

$$\Lambda_k(Z) = \sum_{l=0}^{d-k} (-1)^{l+k} \binom{l+k}{k} c_1(L|_Z)^l \cap \mathcal{M}_{l+k}(Z),$$

for any  $k = 0, \dots, d$ , where  $c_1(L|_Z)$  is the corresponding Chern class. These equalities are in the Chow group of  $Z$  and all the cycles in them actually have support in  $Z_{\text{sing}}$ .

Since at each point in  $Z$  the local L  cycles determine the topology, and hence the homology, of the local Milnor fibre  $F$ , this theorem somehow indicates that the Milnor classes are determined by the vanishing homology of  $Z$ , i.e., by the kernel of the specialization morphism  $H_*(Z_t) \rightarrow H_*(Z)$ , where  $Z_t$  is the complex manifold defined by the intersection of the zero section of the bundle  $L$  with a section near the one that determines  $Z$ , but which is everywhere transversal to the zero section of  $L$ . So Theorem 1 gives a positive answer to a question raised at the introduction in [10] (c.f. [29, 30]).

We also get the corollary below, which extends and strengthens [10, Corollary 5.13] in the hypersurface case:

**Corollary.** *Assume  $M, L$  and  $Z$  are as above and equip  $M$  with a Whitney stratification  $\{Z_\beta\}$  adapted to  $Z$ . Let  $d$  be the dimension of the singular set  $Z_{\text{sing}}$ . Then we have the following equalities of cycles in the Chow group of  $Z$ :*

$$\mathcal{M}_d(Z) = \sum_{S_\beta \subset Z_{\text{sing}}} \mu^\perp(S_\beta) [\overline{S_\beta}] = \sum_{S_\beta \subset Z_{\text{sing}}} \lambda_{S_\beta}^d [\overline{S_\beta}] = (-1)^d \Lambda_d(Z),$$

where the sums run over the strata of dimension  $d$  which are contained in  $Z_{\text{sing}}$ ,  $\mu^\perp(S_\beta)$  is the transversal Milnor number of  $S_\beta$  and  $\lambda_{S_\beta}^d$  is the  $d$ -th L  number of  $S_\beta$ .

The assumption about the bundle  $L$  being very ample is used to have a projective embedding of  $M$  such that  $L$  is the pull back of the tautological

bundle; this leads to a description of the Schwartz-MacPherson classes in terms of polar cycles. The “very ampleness condition” is also needed to use R. Piene’s theorem in [27], expressing the Mather classes in polar terms. Yet, Theorem 1 describes intrinsically the Milnor classes in terms of L  cycles and vice versa, and it does not depend on the projective embedding of  $M$ . This suggests that the very ampleness condition on the line bundle  $L$  may not be necessary.

The trail for getting to Theorem 1 can be roughly described as follows. The first step is defining the global L  cycles. This can be done in various ways. Here we do it by using the interpretation of the local L  cycles given by T. Gaffney and R. Gassler in [16]: We blow up the singular set of  $Z \subset M$  and look at the Chern class of the tautological bundle of the corresponding exceptional divisor. This gives an explicit definition of the global L  cycles.

Next we observe that the main theorem of A. Parusinski and P. Pragacz in [26] expresses the total Milnor class as a function of the Schwartz-MacPherson classes of the closure of the strata of a Whitney stratification (see equation (17)):

$$\mathcal{M}(Z) := \sum_{S_\alpha \in \mathcal{S}} \gamma_{S_\alpha} (c(L|_Z)^{-1} \cap (i_{\bar{S}_\alpha, Z})_* c^{SM}(\bar{S}_\alpha)). \quad (1)$$

We refer to section 4 below for an explanation of the terms involved in this formula. On the other hand, J.-P. Brasselet in [6] conjectured that the Milnor classes can be expressed in terms of polar varieties, which brings us closer to our goal of comparing Milnor classes with L  cycles (which are defined via polar varieties in [22, 23]). Following this path we notice that J. Sch rmann and M. Tib r introduced in [31], in the affine context, the MacPherson cycles associated to any constructible function on a complex algebraic proper subset  $X \subset \mathbb{C}^N$ . They showed that the corresponding cycle class represents the (dual) Schwartz-MacPherson class in the Borel-Moore homology group, and also in the Chow group. We prove an analogous result in Section 3 in the projective case (Theorem 3.7). In this construction a key role is played by certain projective polar varieties. This suffices for us, since the assumption of considering a very ample line bundle  $L$  over  $M$  implies that  $M$  is a projective variety. We prove (see the text for explanations):

**Theorem 2.** *Let  $X$  be an  $n$ -dimensional projective variety endowed with a Whitney stratification with connected strata  $S_\alpha$ . Let  $i_{\bar{S}_\alpha, X} : \bar{S}_\alpha \rightarrow X$  be the natural inclusion. Consider  $\varphi : X \rightarrow \mathbb{C}P^N$  a closed immersion and  $\mathcal{L} = \mathcal{O}_{\mathbb{C}P^N}(1)$ . If  $\beta : X \rightarrow \mathbb{Z}$  is a constructible function with respect to this stratification, then the  $k^{th}$  Schwartz-MacPherson class of  $\beta$ ,  $c_k^{SM}(\beta)$ , is given by:*

$$c_k^{SM}(\beta) = \sum_{\alpha} \eta(S_\alpha, \beta) \sum_{i=k}^{d_\alpha} (-1)^{d_\alpha-i} \binom{i+1}{k+1} (i_{\bar{S}_\alpha, X})_* (c_1(\varphi^* \mathcal{L})^{i-k} \cap [\mathbb{P}_i(\bar{S}_\alpha)]),$$

where  $\eta(S_\alpha, \beta)$  is the normal Morse index.

A key point for proving Theorem 2 is Piene’s characterization in [27, Th or me 3] of the Mather classes via polar varieties.

We remark that there is another formula for the MacPherson classes in terms of polar varieties given in [20]. Yet, the expression we need for proving Theorem 1 is the one given by Theorem 2, because this allows comparison with the Lê cycles.

The final ingredient we need for proving Theorem 1 is Massey's concept of Lê cycles for constructible sheaves via polar varieties (see Remark 3.1). We call these Massey cycles. In Section 3 we extend this concept to the projective setting and we prove a formula comparing the Massey cycles with the MacPherson cycles (Proposition 3.6), analogous to Massey's formula in [24, Theorem 7.5].

Theorem 1 is then proved by considering the formula for the Milnor classes by Parusinski-Pragacz, mentioned above, replacing in it the Schwartz-MacPherson classes of  $X$  by the expression given in Theorem 2, and then using Proposition 3.6 to express the Lê cycles in terms of the MacPherson cycles in the projective setting. This is done in Section 5.

We notice too that the proof of Theorem 1 leads to a description of the Milnor classes in terms of polar cycles, thus answering a question raised by J.-P. Brasselet in [6], which was a motivation for this research (see Remark 5.6).

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## 1 Derived Categories

We assume some basic knowledge on derived categories, hypercohomology and sheaves of vanishing cycles as described in [13].

If  $X$  is a complex analytic space then  $\mathcal{D}_c^b(X)$  denotes the derived category of bounded, constructible complexes of sheaves of  $\mathbb{C}$ -vector spaces on  $X$ . We denote the objects of  $\mathcal{D}_c^b(X)$  by something of the form  $F^\bullet$ . The shifted complex  $F^\bullet[l]$  is defined by  $(F^\bullet[l])^k = F^{l+k}$  and its differential is  $d_{[l]}^k = (-1)^l d^{k+l}$ . The constant sheaf  $\mathbb{C}_X$  on  $X$  induces an object  $\mathbb{C}_X^\bullet \in \mathcal{D}_c^b(X)$  by letting  $\mathbb{C}_X^0 = \mathbb{C}_X$  and  $\mathbb{C}_X^k = 0$  for  $k \neq 0$ .

If  $h : X \rightarrow \mathbb{C}$  is an analytic map and  $F^\bullet \in \mathcal{D}_c^b(X)$ , then we denote the sheaf of vanishing cycles of  $F^\bullet$  with respect to  $h$  by  $\phi_h F^\bullet$ .

For  $F^\bullet \in \mathcal{D}_c^b(X)$  and  $p \in X$ , we denote by  $\mathcal{H}^*(F^\bullet)_p$  the stalk cohomology of  $F^\bullet$  at  $p$ , and by  $\chi(F^\bullet)_p$  its Euler characteristic. That is,

$$\chi(F^\bullet)_p = \sum_k (-1)^k \dim_{\mathbb{C}} \mathcal{H}^k(F^\bullet)_p.$$

We also denote by  $\chi(X, F^\bullet)$  the Euler characteristic of  $X$  with coefficients in

$F^\bullet$ , i.e.,

$$\chi(X, F^\bullet) = \sum_k (-1)^k \dim_{\mathbb{C}} \mathbb{H}^k(X, F^\bullet),$$

where  $\mathbb{H}^*(X, F^\bullet)$  denotes the hypercohomology groups of  $X$  with coefficients in  $F^\bullet$ .

When  $F^\bullet \in \mathcal{D}_c^b(X)$  is  $\mathcal{S}$ -constructible, where  $\mathcal{S}$  is a Whitney stratification of  $X$ , we denote it by  $F^\bullet \in \mathcal{D}_{\mathcal{S}}^b(X)$ . We would like to point out the following result which appears in [13, Theorem 4.1.22]:

$$\chi(X, F^\bullet) = \sum_{S \in \mathcal{S}} \chi(F_S^\bullet) \chi(S), \quad (2)$$

where  $\chi(F_S^\bullet) = \chi(F^\bullet)_p$  for an arbitrary point  $p \in S$ .

For a subvariety  $V$  of  $M$ , we denote its conormal variety by  $T_V^*M$ . That is,

$$T_V^*M := \text{closure } \{(x, \theta) \in T^*M \mid x \in V_{\text{reg}} \text{ and } \theta|_{T_x V_{\text{reg}}} \equiv 0\},$$

where  $T^*M$  is the cotangent bundle of  $M$  and  $V_{\text{reg}}$  is the regular part of  $V$ .

The following definition is standard in the literature:

**Definition 1.1.** Let  $X$  be an analytic (or algebraic) subvariety of a complex manifold  $M$ ,  $\{S_\alpha\}$  a Whitney stratification of  $M$  adapted to  $X$  and  $x \in S_\alpha$  a point in  $X$ . Consider  $g : (M, x) \rightarrow (\mathbb{C}, 0)$  a germ of holomorphic function such that  $d_x g$  is a *non-degenerate covector* at  $x$  with respect to the fixed stratification, that is,  $d_x g \in T_{S_\alpha}^*M$  and  $d_x g \notin T_{S'}^*M$ , for all stratum  $S' \neq S_\alpha$ . And let  $N$  be a germ of a closed complex submanifold of  $M$  which is transversal to  $S_\alpha$ , with  $N \cap S_\alpha = \{x\}$ . Define the *complex link*  $l_{S_\alpha}$  of  $S_\alpha$  by:

$$l_{S_\alpha} := X \cap N \cap B_\delta(x) \cap \{g = w\} \quad \text{for } 0 < |w| \ll \delta \ll 1.$$

The *normal Morse datum* of  $S_\alpha$  is defined by:

$$NMD(S_\alpha) := (X \cap N \cap B_\delta(x), l_{S_\alpha}),$$

and the *normal Morse index*  $\eta(S_\alpha, F^\bullet)$  of the stratum is:

$$\eta(S_\alpha, F^\bullet) := \chi(NMD(S), F^\bullet),$$

where the right-hand-side means the Euler characteristic of the relative hypercohomology.

By a result of M. Goresky and R. MacPherson in [17, Theorem 2.3] we get that the number  $\eta(S_\alpha, F^\bullet)$  does not depends on the choices of  $x \in S_\alpha$ ,  $g$  and  $N$ .

Notice that by [13, Remark 2.4.5(ii)], it follows that

$$\eta(S_\alpha, F^\bullet) = \chi(X \cap N \cap B_\delta(x), F^\bullet) - \chi(l_{S_\alpha}, F^\bullet). \quad (3)$$

**Lemma 1.2.** *Let  $F^\bullet \in \mathcal{D}_S^b(X)$  with  $\mathcal{S} = \{S_\alpha\}$  a Whitney stratification of  $X$  and  $g : X \rightarrow \mathbb{C}$  be a non-degenerate covector at  $p \in S_\alpha$  with respect to the fixed stratification. Set  $d = \dim X$ ,  $d_\alpha = \dim S_\alpha$  and  $m_\alpha := (-1)^{d-d_\alpha-1} \chi(\phi_{g|_N} F^\bullet_{|_N})_p$ , where  $\phi_{g|_N} F^\bullet_{|_N}$  is the sheaf of vanishing cycles of  $F^\bullet_{|_N}$  with respect to  $g|_N$ ,  $p \in S_\alpha$  and  $N$  is a germ of a closed complex submanifold which is transversal to  $S_\alpha$  with  $N \cap S_\alpha = \{p\}$ . Then*

$$m_\alpha = (-1)^{d-d_\alpha} \eta(S_\alpha, F^\bullet).$$

*Proof.* By [13, Equation (4.1), p. 106] we have that

$$\mathcal{H}^i(\phi_g F^\bullet)_p \simeq \mathbb{H}^{i+1}(B_\epsilon(p) \cap X, B_\epsilon(p) \cap X \cap g^{-1}(\varsigma), F^\bullet), \text{ for } 0 < |\varsigma| \ll \epsilon \ll 1.$$

Hence

$$\chi(\phi_{g|_N} F^\bullet_{|_N})_p = -\chi(B_\epsilon(p) \cap X \cap N, B_\epsilon(p) \cap X \cap N \cap g^{-1}(\varsigma), F^\bullet),$$

and therefore  $m_\alpha = (-1)^{d-d_\alpha} \eta(S_\alpha, F^\bullet)$ .  $\square$

**Remark 1.3.** Everything we have defined so far for a constructible complex of sheaves is defined by J. Schürmann and M. Tibăr in [31] for constructible functions, and the two constructions are somehow equivalent. In fact, given  $F^\bullet \in \mathcal{D}_c^b(X)$ , we have naturally associated the constructible function on  $X$  given by

$$\beta(p) = \chi(F^\bullet)_p.$$

Moreover, by Schürmann [30], the converse also holds, *i.e.*, given any constructible function  $\beta$  on  $X$  there is  $F^\bullet \in \mathcal{D}_c^b(X)$  such that

$$\beta(p) = \chi(F^\bullet)_p.$$

The next result follows by Lemma 1.2 and [23, Remark 10.5].

**Corollary 1.4.** *Let  $h : X \rightarrow \mathbb{C}$  be an analytic map. If  $P^\bullet = (\phi_h \mathbb{C}_X^\bullet)[n-d]$  then  $m_\alpha = (-1)^{d-d_\alpha} \eta(S_\alpha, P^\bullet) = (-1)^{d-d_\alpha} \eta(S_\alpha, w)$ , where  $w$  is the constructible function defined by  $w(p) = \chi(P^\bullet)_p = \chi(F_{h,p}) - 1$  with  $F_{h,p}$  being the Milnor fiber of  $h$  at  $p$ .*

## 2 L $\hat{e}$ cycles

Let us recall first the definition of L $\hat{e}$  cycles and L $\hat{e}$  numbers of germs of complex analytic functions introduced by D. Massey in [22] (see also [23]). Let  $U$  be an open subset of  $\mathbb{C}^{n+1}$  containing the origin,  $h : (U, 0) \rightarrow (\mathbb{C}, 0)$  the germ of an analytic function,  $\Sigma(h)$  the critical set of  $h$ , and  $z = (z_0, \dots, z_n)$  a linear choice of coordinates in  $\mathbb{C}^{n+1}$ . To define the L $\hat{e}$  cycles we first need to define the relative polar cycles, which are associated to the relative polar varieties: For each  $k$  with  $0 < k < n$ , the polar variety  $\Gamma_{h,z}^k$  is the scheme  $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n}\right) / \Sigma(h)$ . At the level of ideals, this is generated by the elements

in the ideal of  $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n}\right)$  which are not contained in that of  $\Sigma(h)$ . Massey denotes by  $[\Gamma_{h,z}^k]$  the cycle associated with this scheme (see [23, p. 9]).

Then, for each  $0 < k < n$ , Massey defines the  $k$ -th L  cycle  $\Lambda_{h,z}^k$  of  $h$  with respect to the coordinate system  $z$  as the cycle:

$$\Lambda_{h,z}^k := \left[ \Gamma_{h,z}^{k+1} \cap V\left(\frac{\partial h}{\partial z_k}\right) \right] - [\Gamma_{h,z}^k].$$

If a point  $p = (p_0, \dots, p_n) \in U$  is an isolated point of the intersection of  $\Lambda_{h,z}^k$  with the cycle of  $V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})$ , then the L  number  $\lambda_{h,z}^k(p)$  is the intersection number at  $p$ :

$$\lambda_{h,z}^k(p) := (\Lambda_{h,z}^k \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1}))_p.$$

It is proved in [24, Theorem 7.5] (see also [23, Theorem 10.18]) that for a generic choice of coordinates, all the L  numbers of  $h$  at  $p$  are defined and they are independent of the choice of coordinates. Hence these are called the (*generic*) *L  numbers* of  $h$  at  $p$  and they are denoted simply by  $\lambda_h^k(p)$ .

Massey gives an alternative characterization of the L  cycles of a hypersurface singularity, which leads to a generalization of the L  numbers that can be applied to any constructible sheaf complexes. From this more general viewpoint, the case of the L  numbers of a function  $h$  is just the case where the underlying constructible sheaf complexes is the sheaf of vanishing cycles along  $h$ . Let us explain this.

Let  $X$  be an analytic germ of a  $d$ -dimensional space which is embedded in some affine space,  $M := \mathbb{C}^{n+1}$ , so that the origin is a point of  $X$ . Consider a bounded, constructible sheaf  $F^\bullet$  on  $X$  or  $M$ .

For a generic linear choice of coordinates  $z = (z_0, \dots, z_n)$  for  $\mathbb{C}^{n+1}$ , Massey in [24, Proposition 0.1] proves that there exist analytic cycles  $\Lambda_i(F^\bullet)$  in  $X$  which are purely  $i$ -dimensional, such that  $\Lambda_i(F^\bullet)$  and  $V(z_0 - x_0, \dots, z_{i-1} - x_{i-1})$  intersect properly at each point  $x \in X$  near the origin, and such that for all  $x$  near the origin in  $X$  we have:

$$\chi(F^\bullet)_x = \sum_{i=0}^d (-1)^{d-i} (\Lambda_i(F^\bullet) \cdot V(z_0 - x_0, \dots, z_{i-1} - x_{i-1}))_x. \quad (4)$$

Moreover, whenever such analytic cycles  $\Lambda_i(F^\bullet)$  exist, they are unique. He also sets  $\lambda_{F^\bullet}^i(x) = (\Lambda_i(F^\bullet) \cdot V(z_0 - x_0, \dots, z_{i-1} - x_{i-1}))_x$  and calls it the  $i$ -th characteristic polar multiplicity of  $F^\bullet$ .

When  $\beta(p) = \chi(F^\bullet)_p$  we also denote  $\Lambda_i(F^\bullet)$  by  $\Lambda^i(\beta)$ .

**Remark 2.1.** By [22, Remark 10.5] it follows that if we have  $h : (U, 0) \rightarrow (\mathbb{C}, 0)$  with  $U$  an open neighborhood of the origin in  $\mathbb{C}^{n+1}$ ,  $X = \Sigma(h)$  the critical set of  $h$ ,  $d = \dim_0 X$  and we let

$$P^\bullet = (\phi_h \mathbb{C}_U^\bullet)_{|\Sigma(h)}[n-d],$$

then for generic  $z$ , for all  $i$  and for all  $x \in \Sigma(h)$  near the origin, we have  $\Lambda_i(P^\bullet) = \Lambda_h^i$  and  $\lambda_{P^\bullet}^i(x) = \lambda_h^i(x)$ .



The work of T. Gaffney and R. Gassler [16] describes the generic L   cycles in a coordinate-free way (see also [24, Corollary 7.9]). For this, consider the blow up  $\tilde{U}$  of  $U$  along the variety  $\Sigma(h)$ , with exceptional divisor  $D$ . One has a diagram:

$$\begin{array}{ccc} D & \hookrightarrow & \tilde{U} \\ \downarrow & & \downarrow b \\ \Sigma(h) & \hookrightarrow & U \end{array}$$

Let  $\mathcal{L}$  be the tautological line bundle of  $\tilde{U}$  and  $c_1(\mathcal{L}) \in H^2(\tilde{U})$  its Chern class. Gaffney and Gassler prove that one has:

$$\Lambda_h^k = b_* (c_1(\mathcal{L})^{n-k} \cap [D]).$$

Notice that this equality is as classes in the appropriate Chow group. For simplicity, we denote by the same symbols the cycles and their associated classes in the Chow group.

Of course these definitions and results extend naturally to map-germs defined on open sets in complex manifolds. Moreover, the aforementioned interpretation of the generic L   cycles, due to Gaffney and Gassler, leads naturally to the following notions of L   cycles associated to global hypersurfaces in complex manifolds.

Let  $M$  be an  $n + 1$ -dimensional compact complex analytic manifold, and let  $(L, M, \pi)$  be a holomorphic line bundle over  $M$ . Let  $s$  be a holomorphic section of  $L$ , generically transverse to the zero-section of  $L$ , and set  $Z := s^{-1}(0)$ . That is,  $Z$  is the divisor of zeros of this section, which is a codimension one subspace of  $M$ . Notice that  $Z$  is locally defined by the set of zeros of a holomorphic function, hence we call  $Z$  a global hypersurface of  $M$ . This does not mean that  $Z$  is defined by a global equation. Let  $Z_{\text{sing}}$  be the singular set of  $Z$ . This is the set of points where the section  $s$  fails to be transversal to the zero-section of  $L$ . Consider the blow up  $Bl_{Z_{\text{sing}}} M = B$  of  $M$  along  $Z_{\text{sing}}$ , let  $D$  be the exceptional divisor of  $B$  and  $\mathcal{L}$  the associated line bundle on  $B$ , that we call the *tautological line bundle* of  $B$ . One has a diagram:

$$\begin{array}{ccc} & & \mathcal{L} \\ & & \downarrow \\ D & \hookrightarrow & Bl_{Z_{\text{sing}}} M = B \\ \downarrow & & \downarrow b \\ Z_{\text{sing}} & \hookrightarrow & M \end{array}$$

**Definition 2.2.** For  $0 \leq k \leq n$ , we define the  $k$ -th L   cycle of the section  $s$  as the following class in the Chow group of  $M$ :

$$\Lambda_k(Z) = b_* (c_1(\mathcal{L})^{n-k} \cap [D]).$$

Notice that  $\Lambda_k(Z)$  is supported in  $Z_{\text{sing}}$ , hence we will also look at  $\Lambda_k(Z)$  as a class in the Chow group of  $Z_{\text{sing}}$ . Thus we have that  $\Lambda_k(Z)$  is zero for  $k > \dim Z_{\text{sing}}$ .

We point out that the cycles obtained in this way are special cases of *Segre cycles* (see for instance [14]). Moreover, because  $M$  is compact, these represent classes in the homology ring  $H_*(M; \mathbb{Z})$ .

The above discussion shows that restricted to a coordinate chart we have:

$$\Lambda_k(Z)|_{U_i} = \Lambda_{s_i}^k, \quad (5)$$

as classes in the Chow group of  $U_i$ , where the right-hand side is the class of the generic L   cycle defined by D. Massey and where  $\{(U_i, s_i)\}$  is a local description of  $s$ .

### 3 Schwartz-MacPherson classes

In this section we describe the Schwartz-MacPherson classes of a subvariety of a projective space via projective polar varieties. We recall first the classical construction defined in [21]. Notice that this applies to complex analytic varieties in general, not only for hypersurfaces.

For any subvariety  $X$  of a complex manifold  $M$ , consider the Nash blow up  $\tilde{X} \xrightarrow{\nu} X$  of  $X$ , its Nash bundle  $\tilde{T} \xrightarrow{\pi} \tilde{X}$ , and the Chern classes of  $\tilde{T}$ ,  $c^j(\tilde{T}) \in H^{2j}(\tilde{X})$ ,  $j = 1, \dots, n$ . The Poincar   morphism  $\beta_{\tilde{X}} : H^{2p}(\tilde{X}) \xrightarrow{\cap[\tilde{X}]} H_{2(n+1)-2p}(\tilde{X})$ , carries these into homology classes, which in turn can be pushed forward into the homology of  $X$  via the homomorphism  $\nu_*$  induced by the projection. These are, by definition [21], the Mather classes of  $X$ :

$$c_k^{Ma}(X) := \nu_*(c^{n-k}(\tilde{T}) \cap [\tilde{X}]) \in H_{2k}(X), \quad k = 0, \dots, n.$$

In fact one has a homomorphism  $c_*^{Ma} : Z(M) \rightarrow H_*(M)$  defined in the obvious way, from the free abelian group  $Z(M)$  generated by the irreducible subvarieties of  $M$  to the homology group (or Chow group, or Borel-Moore homology group). We also define the dual Mather class  $\check{c}_*^{Ma}(X)$  of  $X$  by  $\check{c}_k^{Ma}(X) := (-1)^{\dim X - k} c_k^{Ma}(X)$ .

The MacPherson classes are obtained from these by considering appropriate “weights” for each stratum, determined by the local Euler obstruction  $Eu_X(x)$ . This is an integer associated to each point  $x \in X$ . We refer to the literature for the definition of the invariant  $Eu_X(x)$  (see for instance [21, 8] or [11, Ch. 8]).

This invariant is constant in each Whitney stratum (see [21, 8, 20]), so it defines a constructible function on  $X$ . MacPherson shows in [21] that there exists a unique set of integers  $b_\alpha$  for which the equation  $\sum b_\alpha Eu_{\overline{X}_\alpha}(x) = 1$  is satisfied for all points  $x \in X$ , where  $\{X_\alpha\}$  is a Whitney stratification of  $X$ , the sum runs over all strata  $X_\alpha$  containing  $x$  in their closure and  $Eu_{\overline{X}_\alpha}$  is the local Euler obstruction of  $\overline{X}_\alpha$ . Extending this by linearity one gets a homomorphism  $Eu : Z(M) \rightarrow F(M)$  from the free abelian group  $Z(M)$  to the group of constructible functions on  $X$ .

We also define the dual Euler class  $\check{Eu}$  as the transformation which associates to an irreducible analytic subset  $X$  of  $M$  the constructible function

$\tilde{Eu}_X := (-1)^{\dim(X)} Eu_X$ , determined by the local Euler obstruction (see [21] or [18]); this is extended linearly to cycles. Then  $\tilde{Eu}$  is an isomorphism of groups, since  $Eu_X|_{X_{reg}}$  is constant of value 1.

The MacPherson class of degree  $k$  is defined by

$$c_k^M(X) := \sum b_\alpha i_*(c_k^{Ma}(\overline{X}_\alpha)),$$

where  $\{X_\alpha\}$  is a Whitney stratification of  $X$ ,  $\overline{X}_\alpha$  denotes the closure of the stratum, which is itself analytic (and therefore it has its own Mather classes), and  $i : \overline{X}_\alpha \hookrightarrow X$  is the inclusion map. Notice that these are homology classes by definition:  $c_k^M(X) \in H_{2k}(X; \mathbb{X})$ ,  $k = 0, 1, \dots, n-1$ . For non-singular varieties, these are the Poincaré duals of the usual Chern classes.

We notice too that by [8], the MacPherson classes coincide, up to Alexander duality, with the classes defined by M.-H. Schwartz in [32]. Thus, following the modern literature (see for instance [26, 10, 11]), we call these the Schwartz-MacPherson classes of  $X$  and will be denoted from now on by  $c_k^{SM}(X)$ .

Let  $L(M)$  be the free abelian group of all cycles generated by the conormal spaces  $T_X^*M$ , where  $X$  varies over all subvarieties of  $M$ . Define the map  $cn : Z(M) \rightarrow L(M)$  by  $cn(X) := T_X^*M$ . Clearly, this is an isomorphism. We also define the map  $S : L(M) \rightarrow H_*(M)$  by  $S(X) = c^*(T^*M|_X) \cap s_*(T_X^*M)$ , where the Segre class  $s_*$  is defined by

$$s_*(T_X^*M) := \hat{\pi}'_*(c^*(\mathcal{O}(-1))^{-1} \cap [P(T_X^*M)]) = \sum_{i \geq 0} \pi'_*(c^1(\mathcal{O}(1))^i \cap [P(T_X^*M)]),$$

where  $\mathcal{O}(1)$  denotes the tautological line subbundle on the projectivisation  $\hat{\pi}' : P(T^*M|_X) \rightarrow X$  with  $\mathcal{O}(-1)$  as its dual.

By [28, Lemme (1.2.1)] (see also [19, Lemma 1]), one has a description of the dual Mather class of  $X$  in terms of the Segre class of the conormal space  $T_X^*M$ , as follows:

$$\check{c}_*^{Ma}(X) = c^*(T^*M|_X) \cap s_*(T_X^*M). \quad (6)$$

Finally we define a function  $CC : F(M) \rightarrow L(M)$  by

$$CC(\alpha) := \sum_{S \in \mathcal{S}} (-1)^{\dim S} \eta(S, \alpha) \cdot T_S^*M,$$

where  $\alpha$  is a constructible function on  $M$  with respect to the Whitney stratification  $\mathcal{S}$  of  $M$  and  $\eta(S, \alpha)$  is the normal Morse index  $\eta(S, F^\bullet)$ , (see Definition 1.1) where  $F^\bullet$  is the constructible complex of sheaves such that  $\alpha(p) = \chi(F^\bullet)_p$  (see Remark 1.3). One can prove that this map  $CC$  is an isomorphism (see for example the discussion on [31] and the reference therein).

From the above discussions we have the following commutative diagram:

$$\begin{array}{ccccc} F(M) & \xleftarrow{\tilde{Eu}} & Z(M) & \xrightarrow{\check{c}_*^{Ma}} & H_*(M) \\ \downarrow \text{id} & & \downarrow cn & & \downarrow \text{id} \\ F(M) & \xrightarrow{CC} & L(M) & \xrightarrow{S} & H_*(M) \end{array} \quad (7)$$

The commutativity of the left square of this diagram amounts to saying:

$$\beta = \sum_{\alpha} (-1)^{d_{\alpha}} \eta(S_{\alpha}, \beta) \cdot \check{E}u_{S_{\alpha}},$$

for any function  $\beta : X \rightarrow \mathbb{Z}$  which is constructible for the given Whitney stratification. Using the commutativity of the other part of the diagram, we have that

$$\check{c}_k^{SM}(\beta) = \sum_{\alpha} (-1)^{d_{\alpha}} \eta(S_{\alpha}, \beta) (i_{\overline{S}_{\alpha}, X})_* \check{c}_k^{Ma}(\overline{S}_{\alpha}). \quad (8)$$

In the affine context, Schürmann and Tibăr in [31] describe the Schwartz-MacPherson classes of a complex algebraic proper subset  $X \subset \mathbb{C}^N$  using algebraic cycles, which were called MacPherson cycles. In this construction a key role is played by the affine polar varieties, which we now describe.

If  $X$  is of pure dimension  $n < N$ , the  $k$ -th global polar variety of  $X$ ,  $0 \leq k \leq n$ , is the following algebraic set (see [20])

$$P_k(X) = \overline{Crit(x_1, \dots, x_{k+1})|_{X_{reg}}},$$

with  $Crit(x_1, \dots, x_{k+1})|_{X_{reg}}$  the usual critical locus of points  $x \in X_{reg}$  where the differentials of these functions restricted to  $X_{reg}$  are linearly dependent. For general coordinates  $x_i$ , the polar variety  $P_k(X)$  has pure dimension  $k$  or it is empty, for all  $0 < k < n$ . We have  $P_n(X) := X$  and we set  $P_k(X) := \emptyset$  for  $k > n$ .

We fix an algebraic Whitney stratification  $\mathcal{S}$  with connected strata. In this context  $X$  need not be pure dimensional and we only assume  $n = \dim X < N$ . Let  $\alpha : X \rightarrow \mathbb{Z}$  be an  $\mathcal{S}$ -constructible function, meaning that the restriction  $\alpha|_S$  is constant for all strata  $S \in \mathcal{S}$ .

Schürmann and Tibăr define the  $k$ -th MacPherson cycle of  $\alpha$  ( $0 \leq k \leq n$ ) by:

$$MP_k(\alpha) := \sum_{S \in \mathcal{S}} (-1)^{\dim S} \eta(S, \alpha) (\psi_S)_* [P_k(\overline{S})], \quad (9)$$

where  $\psi_S$  is the inclusion of  $\overline{S}$  in  $X$  and  $P_k(\overline{S})$  is the  $k$ -th global polar variety of the algebraic closure  $\overline{S} \subset \mathbb{C}^N$  of the stratum  $S$ .

**Remark 3.1.** Let  $\{S_{\alpha}\}$  be any Whitney stratification of  $X \subset \mathbb{C}^N$  (with connected strata) with respect to which  $F^{\bullet}$  is constructible. Massey obtains an important characterization of the cycles  $\Lambda_i(F^{\bullet})$  using the polar varieties ([23, Corollary 10.15] or [24]). We call the  $k^{th}$  Massey cycle of the constructible sheaf  $F^{\bullet}$  the cycle

$$\Lambda_k(F^{\bullet}) = \sum_{\alpha} m_{\alpha} (\psi_{\alpha})_* [P_k(\overline{S}_{\alpha})], \quad (10)$$

where  $\psi_{\alpha} : \overline{S}_{\alpha} \hookrightarrow X$  is the inclusion,  $m_{\alpha} = (-1)^{d-d_{\alpha}-1} \chi(\phi_{g|_N} F^{\bullet}_{|_N})_p$ , with  $g$  being a non-degenerate covector at  $p \in S_{\alpha}$  with respect to the fixed stratification,  $N$  is a germ of a closed complex submanifold which is transversal to  $S_{\alpha}$  with

$N \cap S_\alpha = \{p\}$ , and  $P_k(\overline{S_\alpha})$  denotes the (affine) polar variety of dimension  $k$  of  $\overline{S_\alpha}$ . Hence, by Lemma 1.2 and Remark 1.3, we have that

$$\Lambda_k(F^\bullet) = (-1)^d MP_k(\beta), \quad (11)$$

where  $\beta(p) = \chi(F^\bullet)_p$  is a constructible function on  $X$  and  $d = \dim X$ .

The most important result of [31] is that the cycle  $MP_k(\alpha)$  represents the  $k$ -th dual Schwartz-MacPherson class  $\check{c}_k^{SM}(\alpha)$  in the Chow group  $A_k(X)$ , where  $\check{c}_k^{SM}(\alpha) = (-1)^k c_k^{SM}(\alpha)$ . That is,

$$c_k^{SM}(\alpha) = (-1)^k MP_k(\alpha) = (-1)^k \sum_{S \in \mathcal{S}} (-1)^{\dim S} \eta(S, \alpha) (\psi_S)_* [P_k(\overline{S})].$$

Hence Schürmann and Tibăr describe in this way the Schwartz-MacPherson classes via affine polar varieties.

These definitions and results motivate the following construction in the projective context.

**Definition 3.2.** Let  $X$  be a complex analytic variety in  $\mathbb{C}P^N$  of pure dimension  $d$ . The  $k$ -th polar variety of  $X$  is defined by

$$\mathbb{P}_k(X) = \overline{\{x \in X_{reg} \mid \dim(T_x X_{reg} \cap L_{k+2}) \geq d - k - 1\}},$$

where  $L_{k+2}$  is a fixed plane of the codimension  $k + 2$  in  $\mathbb{C}P^N$  and  $T_x X_{reg}$  is the projective tangent space of  $X$  at a regular point  $x$ .

Notice that for  $L_{k+2}$  sufficiently general, the dimension of  $\mathbb{P}_k(X)$  in  $X$  is equal to  $k$ . Thus we are indexing the polar varieties by their dimension and not by their codimension, as it is usually done. Observe also that the class  $[\mathbb{P}_k(X)]$  of  $\mathbb{P}_k(X)$  modulo rational equivalence in the Chow group  $A_k(X)$  does not depend on  $L_{k+2}$  provided this is sufficiently general.

**Definition 3.3.** The class  $[\mathbb{P}_k(X)]$  is the  $k$ -th polar class of  $X$ .

For any given  $F^\bullet \in \mathcal{D}_S^b(X)$ , where  $\mathcal{S} = \{S_\alpha\}$  is a Whitney stratification of  $X$ , define the MacPherson cycle

$$MP_k(F^\bullet) := \sum_{\alpha} (-1)^{\dim S_\alpha} \eta(S_\alpha, F^\bullet) (\psi_\alpha)_* [\mathbb{P}_k(\overline{S_\alpha})], \quad (12)$$

where  $\psi_\alpha : \overline{S_\alpha} \hookrightarrow X$  is the inclusion. Sometimes, we also write this cycles by  $MP_k^{\mathbb{P}}(F^\bullet)$  and  $\eta(S_\alpha, F^\bullet)$  by  $\eta^{\mathbb{P}}(S_\alpha, F^\bullet)$ , in order to emphasize the projective nature of this cycle and of this number. If  $\beta$  is the constructible function associated to  $F^\bullet$  as in Remark 1.3 we also denote this cycle  $MP_k^{\mathbb{P}}(F^\bullet)$  by  $MP_k^{\mathbb{P}}(\beta)$ .

Analogously, motivated by equation (10), we define:

**Definition 3.4.** The projective Massey cycle of  $F^\bullet$  is:

$$\Lambda_k^{\mathbb{P}}(F^\bullet) = \sum_{\alpha} m_{\alpha} (\psi_{\alpha})_* [\mathbb{P}_k(\overline{S_\alpha})],$$

where  $\psi_\alpha : \overline{S}_\alpha \hookrightarrow X$  is the inclusion and  $m_\alpha = (-1)^{d-d_\alpha-1} \chi(\phi_{g|_N} F^\bullet|_N)_p$ ;  $g$  is a non-degenerate covector at  $p \in S_\alpha$  with respect to the fixed stratification  $\mathcal{S} = \{S_\alpha\}$ ,  $N$  is a germ of a closed complex submanifold which is transversal to  $S_\alpha$  with  $N \cap S_\alpha = \{p\}$ , and  $\mathbb{P}_k(\overline{S}_\alpha)$  denotes the (projective) polar variety of dimension  $k$  of  $\overline{S}_\alpha$ . When  $\beta(p) = \chi(F^\bullet)_p$  for  $F^\bullet \in \mathcal{D}_S^b(X)$  we denote  $\Lambda_k^\mathbb{P}(F^\bullet)$  also by  $\Lambda_k^\mathbb{P}(\beta)$ .

**Remark 3.5.** We notice that the projective polar variety coincides with the compactification of the affine polar variety. In fact there is a Zariski open dense  $U$  of hyperplanes of  $\mathbb{C}P^N$  such that for all  $H \in U$  we have that

$$\mathbb{P}_k(X) \cap (\mathbb{P}^N \setminus H) = P_k(X \cap (\mathbb{P}^N \setminus H)).$$

Consequently,

$$i^*(MP_k^\mathbb{P}(F^\bullet)) = MP_k(F^\bullet|_{X \cap (\mathbb{P}^N \setminus H)}) \quad \text{and} \quad i^*(\Lambda_k^\mathbb{P}(F^\bullet)) = \Lambda_k(F^\bullet|_{X \cap (\mathbb{P}^N \setminus H)}),$$

where  $i : \mathbb{P}^N \setminus H \hookrightarrow \mathbb{P}^N$  for all  $H \in U' \subset U$  and  $U'$  is such that if  $H \in U'$ ,  $F^\bullet|_{X \cap (\mathbb{P}^N \setminus H)}$  is constructible with respect to the induced Whitney stratification of  $X \setminus H$  with strata

$$\mathcal{S}|_{X \cap (\mathbb{P}^N \setminus H)} := \{S \setminus H \mid S \in \mathcal{S}\}.$$

Hence, for all  $p \in (\mathbb{P}^N \setminus H) \cap X$  we have that  $\eta^\mathbb{P}(S, F^\bullet) = \eta(S, F^\bullet|_{X \cap (\mathbb{P}^N \setminus H)})$ , for all  $S \in \mathcal{S}|_{X \cap (\mathbb{P}^N \setminus H)}$ .

Since two cycles on  $X$  agree if and only if their pull-back to every affine open covering of  $X$  agree, by equation (10), Remark 2.1, and the above discussion we have:

**Proposition 3.6.** *The Massey cycle and the MacPherson cycle agree up to sign. That is:*

$$\Lambda_k^\mathbb{P}(F^\bullet) = (-1)^d MP_k^\mathbb{P}(F^\bullet), \quad (13)$$

where  $d = \dim X$ . And if  $X$  is a hypersurface  $Z$ , then we have:

$$(i_{\text{sing}})_*(\Lambda_k(Z)) = \Lambda_k^\mathbb{P}(w), \quad (14)$$

where  $w$  is the constructible function on  $Z$  defined as in Corollary 1.4 by  $w(p) = \chi(F_{s,p}) - 1$  and  $i_{\text{sing}} : Z_{\text{sing}} \rightarrow Z$  is the natural inclusion.

Notice that this result actually gives a cycle structure to the L  cycle  $\Lambda_k(Z)$ , which is defined only as a class in the Chow group of  $Z_{\text{sing}}$ .

The next result, stated as Theorem 2 in the introduction, is the projective analogous of the formula in [31] describing the Schwartz-MacPherson class in terms of polar cycles.

**Theorem 3.7.** *Let  $X$  be an  $n$ -dimensional projective variety endowed with a Whitney stratification with connected strata  $S_\alpha$ . Let  $i_{\overline{S}_\alpha, X} : \overline{S}_\alpha \rightarrow X$  be the natural inclusion. Consider  $\varphi : X \rightarrow \mathbb{C}P^N$  a closed immersion and  $\mathcal{L} = \mathcal{O}_{\mathbb{C}P^N}(1)$ . If  $\beta : X \rightarrow \mathbb{Z}$  is a constructible function with respect to this stratification, then*

$$\begin{aligned} c_k^{SM}(\beta) &= \sum_{\alpha} \eta(S_\alpha, \beta) \sum_{i=k}^{d_\alpha} (-1)^{d_\alpha-i} \binom{i+1}{k+1} (i_{\overline{S}_\alpha, X})_* (c_1(\varphi^* \mathcal{L})^{i-k} \cap [\mathbb{P}_i(\overline{S}_\alpha)]) \\ &= \sum_{i \geq k} (-1)^i \binom{i+1}{k+1} (c_1(\varphi^* \mathcal{L})^{i-k} \cap MP_i^{\mathbb{P}}(\beta)). \end{aligned}$$

*Proof.* For any purely dimensional projective variety  $V$  of dimension  $d$  we have, by R. Pien's work [27], the following characterization of the Mather classes via polar varieties:

$$c_k^{Ma}(V) = \sum_{i=k}^n (-1)^{n-i} \binom{i+1}{k+1} (c_1(\varphi^* \mathcal{L})^{i-k} \cap [\mathbb{P}_i(V)]). \quad (15)$$

By equation (8) we have that

$$\check{c}_k^{SM}(\beta) = \sum_{\alpha} (-1)^{d_\alpha} \eta(S_\alpha, \beta) (i_{\overline{S}_\alpha, X})_* \check{c}_k^{Ma}(\overline{S}_\alpha). \quad (16)$$

Therefore by equations (15), (16) and the relationship between the MacPherson class and its dual class,

$$\begin{aligned} c_k^{SM}(\beta) &= \sum_{\alpha} \eta(S_\alpha, \beta) \sum_{i=k}^{d_\alpha} (-1)^{d_\alpha-i} \binom{i+1}{k+1} (i_{\overline{S}_\alpha, X})_* (c_1(\varphi^* \mathcal{L})^{i-k} \cap [\mathbb{P}_i(\overline{S}_\alpha)]) \\ &= \sum_{i \geq k} (-1)^i \binom{i+1}{k+1} (c_1(\varphi^* \mathcal{L})^{i-k} \cap MP_i^{\mathbb{P}}(\beta)). \end{aligned}$$

□

## 4 Milnor classes

Let  $M$  be as before, a compact complex manifold of dimension  $n+1$ . We consider a singular hypersurface  $Z$  defined by a holomorphic section  $s$  of some line bundle  $L$  over  $M$ . The Milnor classes of  $Z$  are defined as the difference between the Schwartz-MacPherson classes and the Fulton-Johnson classes of  $Z$ , as we explain below.

The Schwartz-MacPherson classes were described in the previous section, and they provide an extension for singular varieties of the classical Chern classes of complex manifolds. There is another natural generalisation of the Chern classes due to W. Fulton and K. Johnson in [15]. In the case we consider here,

where  $Z$  is a hypersurface, this uses the virtual tangent bundle  $\tau(Z)$  of  $Z$ , which plays the role of the tangent bundle. This virtual bundle is by definition:

$$\tau(Z) := TM|_Z - L|_Z,$$

where  $TM$  denotes the tangent bundle of  $M$ ,  $L$  is the bundle of  $Z$  and the difference is in the KU-theory of  $Z$ . Notice that restricted to the regular part of  $Z$ , the bundle  $L$  is isomorphic to the normal bundle of  $Z$  in  $M$ . The Chern classes of the virtual bundle  $\tau(Z)$  live in the cohomology of  $Z$ , and the Poincaré morphism carries them into homology classes:

$$c^{FJ}(Z) := c(TM|_Z - L|_Z) \cap [Z],$$

where the total Chern class of  $(TM|_Z - L|_Z)$  is defined by  $c(TM|_Z - L|_Z) = i^*(c(TM) \cdot c(L)^{-1})$ , with  $i : Z \hookrightarrow M$  being the inclusion.

**Definition 4.1.** ([2, 9, 10, 36, 26]) The *total Milnor class* of  $Z$  is defined by:

$$\mathcal{M}(Z) := (-1)^{n-1} (c^{FJ}(Z) - c^{SM}(Z)).$$

We denote by  $\mathcal{M}_k(Z)$  the component of this total class in  $H_{2k}(Z)$ ,  $k = 0, \dots, 2n$ , and call it *the Milnor class of  $Z$  of degree  $k$* .

Milnor classes were originally defined as elements in the singular homology group of the hypersurface (or a complete intersection). It was observed in the work of W. Fulton and K. Johnson ([15]) and G. Kennedy ([19]) that the Fulton-Johnson classes and the Schwartz-MacPherson classes, and hence the Milnor classes, can actually be considered as cycle classes in the corresponding Chow group.

In the sequel we will use the following important characterization of Milnor classes obtained by A. Parusinski and P. Pragacz:

$$\mathcal{M}(Z) := \sum_{S \in \mathcal{S}} \gamma_S \left( c(L|_Z)^{-1} \cap (i_{\overline{S}, Z})_* c^{SM}(\overline{S}) \right), \quad (17)$$

where  $\mathcal{S} = \{S\}$  is an analytic Whitney stratification of  $Z$  with connected strata, such that  $Z_{\text{sing}}$  is union of strata,  $i_{\overline{S}, Z} : \overline{S} \hookrightarrow Z$  is the inclusion and  $\gamma_S$  is the function defined on each stratum  $S$  as follows: For each  $x \in S \subset Z$ , let  $F_x$  be a *local Milnor fibre* (recall  $Z$  is a hypersurface in the complex manifold  $M$ ), and let  $\chi(F_x)$  be its Euler characteristic. We set:

$$\mu(x; Z) := (-1)^n (\chi(F_x) - 1),$$

and call it the *local Milnor number* of  $Z$  at  $x$ . This number is constant on each Whitney stratum, by the topological triviality of Whitney stratifications, so we denote it by  $\mu_S$ . Then  $\gamma_S$  is defined inductively by:

$$\gamma_S = \mu_S - \sum_{S' \neq S, \overline{S'} \supset S} \gamma_{S'}.$$



For instance, suppose that the singular set of  $Z$  is a finite number of points  $x_1, \dots, x_r$ , and let  $\mu_{x_i}$  denote the corresponding Milnor numbers. Then by [33] we have  $\mathcal{M}_0(Z) = \sum_{i=1}^r \mu_{x_i} [x_i] \in H_0(Z)$ , and all other Milnor classes are 0 by [34]. More generally one has the following result [9, 10, Theorem 5.2 and Corollary 5.13], that we prove here for completeness, using the description of Milnor classes by A. Parusinski and P. Pragacz (see equation (17)).

**Proposition 4.2.** *The Milnor classes have support in the singular set  $Z_{\text{sing}}$ . That is,*

$$\mathcal{M}(Z) = \sum_{S \subset Z_{\text{sing}}} \gamma_S (c(L|_Z)^{-1} \cap (i_{\bar{S}, Z})_* c^{SM}(\bar{S})).$$

Hence these classes are all zero in dimensions above the dimension of  $Z_{\text{sing}}$ .

*Proof.* Let  $\mathcal{S} = \{S\}$  be an analytic Whitney stratification of  $Z$  with connected strata, such that  $Z_{\text{sing}}$  is union of strata. Notice that the closure  $\bar{S}$  of each stratum is again analytic, so it has its own Schwartz-MacPherson classes  $c^{SM}(\bar{S})$ . By equation (17) we have that

$$\mathcal{M}(Z) := \sum_{S \in \mathcal{S}} \gamma_S (c(L|_Z)^{-1} \cap (i_{\bar{S}, Z})_* c^{SM}(\bar{S})).$$

If  $x$  is a regular point of  $Z$ , then  $\chi(F_x) = 1$  and  $\mu_S = 0$ . Hence

$$\mathcal{M}(Z) = \sum_{S \subset Z_{\text{sing}}} \gamma_S (c(L|_Z)^{-1} \cap (i_{\bar{S}, Z})_* c^{SM}(\bar{S})).$$

Since for strata in the singular set of  $Z$  one has  $\dim \bar{S} \leq \dim(Z_{\text{sing}})$ , then  $c_k^{SM}(S) = 0$  if  $k > \dim(Z_{\text{sing}})$ . Therefore  $\mathcal{M}_k(Z) = 0$  if  $k > \dim(Z_{\text{sing}})$ .  $\square$

Thus, if the dimension of  $Z_{\text{sing}}$  is  $d$ , then we have Milnor classes  $\mathcal{M}_k(Z) \in H_{2k}(Z)$ ,  $k = 0, \dots, d$ , which are non-zero generally speaking, and each has support in  $Z_{\text{sing}}$ .

## 5 Milnor classes via Lê cycles

This section is devoted to proving Theorem 1. We first prove:

**Lemma 5.1 (Main Lemma).** *Let  $M^{n+1}$  be a compact complex manifold, consider a very ample line bundle  $L$  on  $M$  and let  $Z$  be the zero set of a holomorphic section  $s$  of  $L$ . Then the corresponding Milnor classes and Lê cycles satisfy:*

$$\mathcal{M}_k(Z) = \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(L|_Z)^{i-k} \cap \Lambda_i(Z).$$

*This equality is in the Chow group of  $Z$  and, furthermore, all cycles in this formula actually have support in the singular set of  $Z$ .*

*Proof.* Let  $\mathcal{S} = \{S_\alpha\}$  be a Whitney stratification of  $Z$  such that each stratum is connected and  $Z_{\text{sing}}$  is a union of strata.

Since  $L$  is very ample, there is a closed immersion  $f : M \hookrightarrow \mathbb{C}P^N$  such that  $L = f^*(\mathcal{O}_{\mathbb{C}P^N}(1))$ , where  $\mathcal{O}_{\mathbb{C}P^N}(1)$  is the tautological bundle of  $\mathbb{C}P^N$ . Set  $\mathcal{L} = \mathcal{O}_{\mathbb{C}P^N}(1)$ .

Consider the following inclusions and the closed immersion  $f$ :

$$\overline{S}_\beta \xrightarrow{i_{\overline{S}_\beta, \overline{S}_\alpha}} \overline{S}_\alpha \xrightarrow{\psi_\alpha} Z \xrightarrow{g} M \xrightarrow{f} \mathbb{C}P^N.$$

Since the mappings are closed (and therefore proper) and  $\mathbb{C}P^n$  is a compact complex manifold, we have the following characterization of the Milnor classes of  $Z$  given by A. Parusinski and P. Pragacz:

$$\mathcal{M}_k(Z) = \sum_{\alpha} \gamma(S_\alpha) \left( \sum_{j=0}^{d_\alpha-k} (-1)^j c_1(g^* f^* \mathcal{L})^j \cap (\psi_\alpha)_* c_{k+j}^{SM}(\overline{S}_\alpha) \right) \in A_k(Z), \quad (18)$$

where

$$\gamma(S_\alpha) = \mu_{S_\alpha} - \sum_{S_\beta \neq S_\alpha, \overline{S}_\beta \supset S_\alpha} \gamma(S_\beta),$$

with  $\mu_{S_\alpha} = (-1)^n (\chi(F_{Z,[p]}) - 1)$  for some  $[p] \in S_\alpha$ . In fact, equation (18) can be easily obtained from equation (17) by concentrating in degree  $k$  in the Chow group  $A_*(Z)$ .

Using Theorem 3.7 for  $X = \overline{S}_\alpha$  we have that the  $k$ -th Schwartz-MacPherson class,  $c_k^{SM}(\overline{S}_\alpha)$ , is given by

$$\sum_{S_\beta \subset \overline{S}_\alpha} \eta(S_\beta, 1_{\overline{S}_\alpha}) \sum_{i=k+j}^{d_\beta} (-1)^{d_\beta-i} \binom{i+1}{k+j+1} (i_{\overline{S}_\beta, \overline{S}_\alpha})_* \left( c_1(\varphi_\beta^* \mathcal{L})^{i-k-j} \cap [\mathbb{P}_i(\overline{S}_\beta)] \right),$$

where  $\varphi_\beta = f \circ g \circ \psi_\alpha \circ i_{\overline{S}_\beta, \overline{S}_\alpha}$ . Applying this in equation (18), we have that

$$\begin{aligned} \mathcal{M}_k(Z) &= \sum_{\alpha} \gamma(S_\alpha) \sum_{j=0}^{d_\alpha-k} (-1)^j c_1(g^* f^* \mathcal{L})^j \cap (\psi_\alpha)_* \left( \sum_{S_\beta \subset \overline{S}_\alpha} \eta(S_\beta, 1_{\overline{S}_\alpha}) \right. \\ &\quad \left. \sum_{i=k+j}^{d_\beta} (-1)^{d_\beta-i} \binom{i+1}{k+j+1} (i_{\overline{S}_\beta, \overline{S}_\alpha})_* \left( c_1(\varphi_\beta^* \mathcal{L})^{i-k-j} \cap [\mathbb{P}_i(\overline{S}_\beta)] \right) \right). \end{aligned}$$

Using:

- That for all  $i, j \in \{\alpha\}$  we have  $\eta(S_i, 1_{\overline{S}_j}) = 0$  if  $S_i \not\subset \overline{S}_j$ ;
- that by a property of the Euler characteristic we have:

$$\sum_{\beta} \gamma(S_\beta) \eta(S_\alpha, 1_{\overline{S}_\beta}) = \eta \left( S_\alpha, \sum_{\beta} \gamma(S_\beta) 1_{\overline{S}_\beta} \right);$$

- and that by [26, Lemma 4.1]) we have  $\sum_{\beta} \gamma(S_{\beta}) 1_{\overline{S_{\beta}}} = (-1)^n w$ , where

$$w(x) = \chi(F_x) - 1;$$

we get:

$$\begin{aligned} \mathcal{M}_k(Z) &= (-1)^n \sum_{j \geq 0} (-1)^j c_1(g^* f^* \mathcal{L})^j \cap \sum_{\beta} (-1)^{d_{\beta}} \eta(S_{\beta}, w) \\ &\quad \sum_{i \geq k+j} (-1)^i \binom{i+1}{k+j+1} (\psi_{\alpha})_* (i_{\overline{S_{\beta}}, \overline{S_{\alpha}}})_* \left( c_1(\varphi_{\beta}^* \mathcal{L})^{i-k-j} \cap [\mathbb{P}_i(\overline{S_{\beta}})] \right). \end{aligned}$$

That is,

$$\begin{aligned} \mathcal{M}_k(Z) &= (-1)^n \sum_{j \geq 0} (-1)^j c_1(g^* f^* \mathcal{L})^j \cap \sum_{\beta} (-1)^{d_{\beta}} \eta(S_{\beta}, w) \sum_{i \geq k+j} (-1)^i \\ &\quad \binom{i+1}{k+j+1} (\psi_{\beta})_* (c_1(\psi_{\beta})^* g^* f^* \mathcal{L})^{i-k-j} \cap [\mathbb{P}_i(\overline{S_{\beta}})]). \end{aligned}$$

Since

$$(\psi_{\beta})_* (c_1((\psi_{\beta})^* g^* f^* \mathcal{L})^{i-k-j} \cap [\mathbb{P}_i(\overline{S_{\beta}})]) = c_1(g^* f^* \mathcal{L})^{i-k-j} \cap (\psi_{\beta})_* [\mathbb{P}_i(\overline{S_{\beta}})],$$

we have that

$$\begin{aligned} \mathcal{M}_k(Z) &= (-1)^n \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(g^* f^* \mathcal{L})^{i-k} \cap \\ &\quad \cap \sum_{\beta} (-1)^{d_{\beta}} \eta(S_{\beta}, w) (\psi_{\beta})_* [\mathbb{P}_i(\overline{S_{\beta}})] \\ &= (-1)^n \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(g^* f^* \mathcal{L})^{i-k} \cap MP_i^{\mathbb{P}}(w). \end{aligned}$$

By equation (13), since  $L = f^* \mathcal{L}$ , we get:

$$\mathcal{M}_k(Z) = (-1)^n \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(g^* L)^{i-k} \cap ((-1)^n \Lambda_i^{\mathbb{P}}(w)).$$

Hence by equation (14) we have:

$$\mathcal{M}_k(Z) = (-1)^n \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(g^* L)^{i-k} \cap ((-1)^n (i_{sing})_* (\Lambda_i(Z))).$$

Therefore,

$$\mathcal{M}_k(Z) = \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(L|Z)^{i-k} \cap (i_{sing})_* \Lambda_i(Z),$$

as stated.  $\square$

Next we have the following consequence of Lemma 5.1 which proves the first statement in Theorem 1.

**Lemma 5.2.** *The following equality holds in the Chow group of  $Z$ :*

$$\mathcal{M}_k(Z) = \sum_{l=0}^{d-k} (-1)^{k+l} \binom{l+k}{k} c_1(L|_Z)^l \cap \Lambda_{l+k}(Z),$$

where  $d = \dim(Z_{\text{sing}})$ .

*Proof.* We use the following identity:

$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}. \quad (19)$$

We know:

$$\mathcal{M}_k(Z) = \sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(L|_Z)^{i-k} \cap \Lambda_i(Z).$$

Making  $l = i - k$  we get:

$$\mathcal{M}_k(Z) = \sum_{j=0}^{d-k} \sum_{l=j}^{d-k} (-1)^{k+j+l} \binom{l+k+1}{j+k+1} c_1(L|_Z)^l \cap \Lambda_{l+k}(Z).$$

Re-arranging this formula we obtain:

$$\mathcal{M}_k(Z) = \sum_{l=0}^{d-k} \left[ \sum_{j=0}^l (-1)^{k+j+l} \binom{l+k+1}{j+k+1} \right] c_1(L|_Z)^l \cap \Lambda_{l+k}(Z).$$

Hence we only need to show:

$$\sum_{j=0}^l (-1)^j \binom{l+k+1}{j+k+1} = \binom{l+k}{k}.$$

For this we use equation (19); we get:

$$\begin{aligned} \sum_{j=0}^l (-1)^j \binom{l+k+1}{j+k+1} &= \sum_{j=0}^l (-1)^j \left[ \binom{l+k}{j+k} + \binom{l+k}{j+k+1} \right] \\ &= \sum_{j=0}^l (-1)^j \binom{l+k}{j+k} + \sum_{j=0}^{l-1} (-1)^j \binom{l+k}{j+k+1} \\ &= \sum_{j=0}^l (-1)^j \binom{l+k}{j+k} - \sum_{j=1}^l (-1)^j \binom{l+k}{j+k} \\ &= \binom{l+k}{k}, \end{aligned}$$

which completes the proof of Lemma 5.2.  $\square$

In order to prove the second statement in Theorem 1 we need to formulate some technical results.

**Remark 5.3.** Notice that:

$$\sum_{k=m}^n (-1)^k \binom{n}{k} \binom{k}{m} = \sum_{k=m}^n (-1)^k \binom{n}{n-k+m} \binom{n-k+m}{m}.$$

The following lemma is an immediate consequence of the associative law:

**Lemma 5.4.** *Let  $a_{n,l}$  be elements in an abelian (additive) group. Then,*

$$\sum_{n=q}^d \sum_{l=0}^{d-n} a_{n,l} = \sum_{s=q}^d \sum_{t=0}^{s-q} a_{s-t,t}.$$

We now prove the lemma below, which completes the proof of Theorem 1.

**Lemma 5.5.** *We have the following equality in the Chow group of  $Z$ :*

$$\Lambda_k(Z) = \sum_{l=0}^{d-k} (-1)^{l+k} \binom{l+k}{k} c_1(L|_Z)^l \cap \mathcal{M}_{l+k}(Z),$$

where  $d = \dim(Z_{\text{sing}})$ .

*Proof.* We will prove that we have:

$$\Lambda_{d-k}(Z) = \sum_{l=0}^k (-1)^{d-k+l} \binom{d-k+l}{d-k} c_1(L|_Z)^l \cap \mathcal{M}_{d-k+l}(Z),$$

which implies the lemma. Let

$$A_{d,k} = \sum_{l=0}^k (-1)^{d-k+l} \binom{d-k+l}{d-k} c_1(L|_Z)^l \cap \mathcal{M}_{d-k+l}(Z).$$

Then, making  $l = k - j$ , we obtain:

$$A_{d,k} = \sum_{j=0}^k (-1)^{d-j} \binom{d-j}{d-k} c_1(L|_Z)^{k-j} \cap \mathcal{M}_{d-j}(Z).$$

Using Lemma 5.2 we obtain:

$$\begin{aligned} A_{d,k} &= \sum_{j=0}^k (-1)^{d-j} \binom{d-j}{d-k} c_1(L|_Z)^{k-j} \cap \\ &\quad \cap \left[ \sum_{l=0}^j (-1)^{d-j+l} \binom{d-j+l}{d-j} c_1(L|_Z)^l \cap \Lambda_{d-j+l}(Z) \right]. \end{aligned}$$

Therefore

$$A_{d,k} = \sum_{j=0}^k \sum_{l=0}^j (-1)^l \binom{d-j+l}{d-j} \binom{d-j}{d-k} c_1(L|_Z)^{k+l-j} \cap \Lambda_{d-j+l}(Z).$$

We now set  $d-j = n$ , so we get:

$$A_{d,k} = \sum_{n=d-k}^d \sum_{l=0}^{d-n} (-1)^l \binom{n+l}{n} \binom{n}{d-k} c_1(L|_Z)^{k+l-d+n} \cap \Lambda_{n+l}(Z).$$

Then Lemma 5.4 implies:

$$A_{d,k} = \sum_{s=d-k}^d \sum_{t=0}^{s-d+k} (-1)^t \binom{s}{s-t} \binom{s-t}{d-k} c_1(L|_Z)^{s-d+k} \cap \Lambda_s(Z).$$

That is,

$$A_{d,k} = \sum_{s=d-k}^d (-1)^s \left[ \sum_{t=0}^{s-d+k} (-1)^{s-t} \binom{s}{s-t} \binom{s-t}{d-k} \right] c_1(L|_Z)^{s-d+k} \cap \Lambda_s(Z).$$

Making  $t = u - d + k$  we obtain that  $A_{d,k}$  equals the sum:

$$\sum_{s=d-k}^d (-1)^s \left[ \sum_{u=d-k}^s (-1)^{s-u+d-k} \binom{s}{s-u+d-k} \binom{s-u+d-k}{d-k} \right] c_1(L)^{s-d+k} \cap \Lambda_s(Z).$$

Hence the previous remark implies:

$$A_{d,k} = \sum_{s=d-k}^d (-1)^s \left[ \sum_{v=d-k}^s (-1)^v \binom{s}{v} \binom{v}{d-k} \right] c_1(L|_Z)^{s-d+k} \cap \Lambda_s(Z).$$

So, using [35, Lemma (2.6)], we get :

$$A_{d,k} = \sum_{s=d-k}^d (-1)^s (-1)^{d-k} \delta_{s,d-k} c_1(L|_Z)^{s-d+k} \cap \Lambda_s(Z).$$

where  $\delta_{m,n}$  as before denotes the Kronecker delta. Therefore

$$A_{d,k} = \Lambda_{d-k}(Z),$$

as stated.  $\square$

**Remark 5.6.** It follows from the above proof that one has the following relations between the Milnor classes and polar cycles:

$$\mathcal{M}_k(Z) = \sum_{l=0}^{d-k} (-1)^{n+k+l} \binom{l+k}{k} c_1(L)^l \cap \sum_{\alpha} (-1)^{d_{\alpha}} \eta(S_{\alpha}, w) [\mathbb{P}_{l+k}(\overline{S_{\alpha}})],$$

answering in this way a question raised by J.-P. Brasselet in [6].

In a more general context in which  $X$  is embedded as a closed subscheme of a smooth scheme  $M$ , P. Aluffi in [3] introduces a cycle  $\alpha_X$  in the Chow group, called later the Aluffi class, which is related to the Donaldson-Thomas type invariants described by K. Behrend in [5].

In the case we envisage here  $X$  is the singular set  $Z_{\text{sing}}$  of a hypersurface  $Z$  in a  $n + 1$ -dimensional manifold  $M$ . We have that  $\alpha_{Z_{\text{sing}}} = c^{SM}(\nu_{Z_{\text{sing}}})$ , with  $\nu_{Z_{\text{sing}}}(p) = (-1)^{n+1}(1 - \chi(F_p))$ , for all  $p \in Z_{\text{sing}}$ . Note that  $\nu_{Z_{\text{sing}}} = (-1)^n w$ , as defined in the Corollary 1.4.

**Corollary 5.7.** *The Aluffi class  $\alpha_{Z_{\text{sing}}}$  of  $Z_{\text{sing}}$  relates to the L  cycles as follows:*

$$(\alpha_{Z_{\text{sing}}})_k = \sum_{l \geq 0} \sum_{m=0}^{d-l-k} (-1)^{m+l+k} \binom{m+l+k}{l+k} c_1(L)^{m+l} \cap \Lambda_{m+l+k}(Z).$$

Hence the Aluffi classes relate to the polar varieties as follows:

$$\begin{aligned} (\alpha_{Z_{\text{sing}}})_k &= \sum_{l \geq 0} \sum_{m=0}^{d-l-k} (-1)^{n+m+l+k} \binom{m+l+k}{l+k} c_1(L)^{m+l} \cap \\ &\cap \left( \sum_{\alpha} (-1)^{d_{\alpha}} \eta(S_{\alpha}, w) [\mathbb{P}_{m+l+k}(\overline{S_{\alpha}})] \right). \end{aligned}$$

*Proof.* Given  $\mathcal{S} = \{S\}$  a Whitney stratification of  $Z$  such that  $Z_{\text{sing}}$  is a union of strata, using [26, Lemma 4.1] we have that

$$\alpha_{Z_{\text{sing}}} = \sum_{S \in \mathcal{S}} \gamma_S (i_{\overline{S}, Z})_* c^{SM}(\overline{S}), \quad (20)$$

where  $\gamma_S = (\nu_{Z_{\text{sing}}})_S - \sum_{S' \neq S, \overline{S'} \supset S} \gamma_{S'}$  and  $i_{\overline{S}, Z} : \overline{S} \hookrightarrow Z$  is the inclusion. Hence

by the description of the Milnor classes given by A. Parusinski and P. Pragacz in [26] (see also equation (17)), we have that

$$\alpha_{Z_{\text{sing}}} = c(L|_Z) \cap \mathcal{M}(Z). \quad (21)$$

Thus the result follows by the Theorem 1. The second expression follows similarly using the description of the Milnor classes via polar varieties as in Remark 5.6.  $\square$

We note that in [3] the author relates the Aluffi classes to the Milnor classes, and formula (21) already appears in that article. Yet, the proof we give here is shorter and more direct (see [3, Theorem 1.2]).

## 6 On the top dimensional Milnor class

Let  $\mathcal{S} = \{S_\alpha\}$  be a Whitney stratification of  $M$  with connected strata, adapted to  $Z$  and to  $Z_{\text{sing}}$ . Set  $d = \dim(Z_{\text{sing}})$  and assume that all  $d$ -dimensional strata of  $\mathcal{S}$  which lie in  $Z_{\text{sing}}$  are contained in the non-singular locus of  $Z_{\text{sing}}$ . Notice that this condition is not automatic unless we demand that the stratification be adapted also to the singular set of  $Z_{\text{sing}}$ . Let  $S_\alpha$  be a  $d$ -dimensional stratum contained in  $Z_{\text{sing}}$ , let  $x$  be a point in  $S_\alpha$ , so it is a regular point of  $Z_{\text{sing}}$ , and let  $H_x$  be a local submanifold of  $M$ , biholomorphic to a disc of complex dimension  $d$ , transversal at  $x$  to the stratum  $S_\alpha$ . Then  $H_x \cap Z$  is a hypersurface in  $H_x$  with an isolated singularity at  $x$ .

**Definition 6.1.** Let  $S_\alpha, x$  and  $H_x$  be as before. The *transversal Milnor number* of  $S_\alpha$  at  $x$  is the usual Milnor number at  $x$  of the isolated hypersurface germ  $H_x \cap Z$ . We denote this number by  $\mu^\perp(S_\alpha, x)$ .

The local topological triviality of Whitney stratifications implies that the transversal Milnor number  $\mu^\perp(S_\alpha, x)$  is constant along  $S_\alpha$ , and of course independent of the choice of the transversal slice  $H_x$ . Hence we denote this number just by  $\mu^\perp(S_\alpha)$  and call it the *transversal Milnor number of  $S_\alpha$* . Notice that the number  $\mu^\perp(S_\alpha)$  is defined only for  $d$ -dimensional strata contained in the regular part of  $Z_{\text{sing}}$ , where  $d = \dim(Z_{\text{sing}})$ . One has:

**Lemma 6.2.** Let  $x \in Z_{\text{sing}}$  be a regular point of the singular set which belongs to a  $d$ -dimensional stratum  $S_x$  of  $Z_{\text{sing}}$ . Set  $\mu(Z, x) = (-1)^n (\chi(F_x) - 1)$ , where  $F_x$  is a local Milnor fibre. Then the transversal Milnor number  $\mu^\perp(S_x)$  equals, up to sign, the local Milnor number  $\mu(Z, x)$  of  $Z$  at  $x$ . More precisely:

$$\mu(Z, x) = (-1)^d \mu^\perp(S_x).$$

*Proof.* By definition we have that

$$\mu^\perp(S_x) = \mu(H_x \cap Z, x) = \mu((s_j)|_{H_x}, x),$$

where  $s_j$  is the restriction  $s|_{U_j}$  of  $s$  to a local chart  $U_j$  around  $x$  in  $Z$ , and  $H_x$  is as above, a local submanifold of  $M$ , biholomorphic to a disc of complex codimension  $d$ , transversal at  $x$  to the stratum  $S_x$ .

Since  $S_x$  and  $Z_{\text{sing}}$  have the same dimension,  $H_x$  is transversal at  $x$  to  $S_x$ . Using the fact that every Whitney stratified set is topologically locally trivial, we have that there exists a local homeomorphism  $(Z, x) \rightarrow (S_x, x) \times (N, x)$ , with  $N = H_x \cap Z$ . Moreover, as every holomorphic function satisfies the Thom condition, we conclude that

$$\chi(F_{s_j, x}) = \chi(F_{(s_j)|_{H_x}, x}) = 1 + (-1)^{n-d} \mu((s_j)|_{H_x}, x),$$

where the last equality follows by the Milnor's Fibration Theorem. Thus

$$\mu(Z, x) = (-1)^n (\chi(F_{s_j, x}) - 1) = (-1)^d \mu((s_j)|_{H_x}, x) = (-1)^d \mu^\perp(S_x).$$

□



**Remark 6.3.** Let  $S$  be a  $d$ -dimensional stratum of a Whitney stratification of  $M$  adapted to  $Z$ , contained in the non-singular locus of  $Z_{\text{sing}}$ . We define the top dimensional Lê number  $\lambda_S^d$  of  $S$  by  $\lambda_S^d := \lambda_{s_i, x}^d$ , where  $\lambda_{s_i, x}^d$  is the Massey Lê number of  $s_i$  at  $x$ , for any  $x \in S \cap U_i$  and with  $U_i$  a local chart around  $x$  in  $Z$  and  $s_i$  the restriction  $s|_{U_i}$ . Then, the transversal Milnor number  $\mu^\perp(S)$  is equal to the top dimensional Lê number  $\lambda_S^d$ . In fact, since we may assume that  $S$  is reduced, irreducible and that for a generic point  $x \in S$  we have that the multiplicity of  $S$  along  $x$  is one, the claim follows directly by the description of the top dimensional Lê number  $\lambda_{s_i}^d(x)$  given by D. Massey in [22, Proposition 2.8] (see also [23, p. 20-21]).

Our last result is the corollary stated in the introduction:

**Corollary 6.4.** *Assume that  $L$  is a very ample line bundle on  $M$ . Then,*

$$\mathcal{M}_d(Z) = \sum_{\substack{S_\alpha \subseteq Z_{\text{sing}} \\ \dim S_\alpha = d}} (-1)^d \mu^\perp(S_\alpha) [\overline{S_\alpha}] = \sum_{\substack{S_\alpha \subseteq Z_{\text{sing}} \\ \dim S_\alpha = d}} (-1)^d \lambda_{S_\alpha}^d [\overline{S_\alpha}], \quad (22)$$

where  $\mu^\perp(S_\alpha)$  is the transversal Milnor number of  $S_\alpha$  and  $\lambda_{S_\alpha}^d$  is the  $d$ -th Lê number of  $S_\alpha$ . This equality is as classes in the Chow group of  $Z$  and hence also in the singular homology group of  $Z$ .

*Proof.* Applying Theorem 1 in the case  $k = \dim Z_{\text{sing}} = d$ , we obtain:

$$\mathcal{M}_d(Z) = (-1)^d \Lambda_s^d,$$

where  $\Lambda_s^d$  can be described by:

$$\sum_{\alpha} m_{\alpha}(\psi_{\alpha})_* [\mathbb{P}_d(\overline{S_{\alpha}})] = \sum_{\dim S_{\alpha} = d} m_{\alpha}(\psi_{\alpha})_* [\overline{S_{\alpha}}],$$

where  $m_{\alpha} := (-1)^{d-d_{\alpha}-1} \chi(\phi_{s_j|_N} F_{|_N}^{\bullet})_x$  with  $d_{\alpha} = \dim S_{\alpha}$  and  $N$  is a local submanifold of  $M$  of complex codimension  $d_{\alpha}$ , transversal at  $x$  to  $S_{\alpha}$ .

For  $x$  in the top dimensional stratum  $S_{\alpha}$ , by [24, Lemma 4.13] we have:

$$(-1)^{n+1-d-1} \chi(\phi_{s_j|_N} F_{|_N}^{\bullet})_x = \mu(s_j|_N) \chi(F_{|_N}^{\bullet})_x = (-1)^{n+1-d} \mu(s_j|_N).$$

Hence  $m_{\alpha} = \mu(s_j|_N) = \mu^\perp(S_{\alpha})$ , which proves the first equality in equation (22). The second equality follows directly by Remark 6.3.  $\square$

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